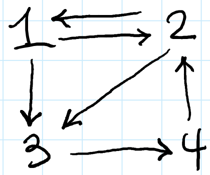
12-Feb

Wednesday, February 12, 2025 10:36 AM

Problem from last time...

ex. consider 4 URLs that may be viewed sequentially in a given pattern :



let x_1 x_2 x_3 x_4 be the proportions of the users that land on each of the pages initially – express as a <u>distribution</u> vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

for example : $\vec{x} = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{bmatrix}$ or $\begin{bmatrix} 2/5 \\ 1/10 \\ 3/10 \\ 1/5 \end{bmatrix}$

40% of users land here (page 1) first

adds to 1 EXACTLY (100%)

users will move from page to page according to system

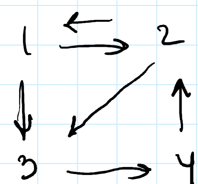
let's write result as transformation : $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

where y_1 = user <u>ends</u> at page 1

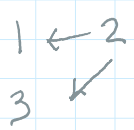
y_2 = " 2

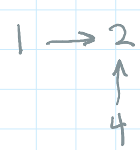
y_3 = " 3

y_4 = " 4



assume equal probability that user moves from one page to the next





$y_1 = \frac{1}{2}x_2$

$y_2 = \frac{1}{2}x_1 \qquad 1x_4$

$y_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2$

$y_4 = \qquad x_3$

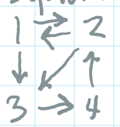
check (add): $1x_1$ $1x_2$ $1x_3$ $1x_4$ ← all coefficients should add to 1 check (add): $1x_1 \quad 1x_2 \quad 1x_3 \quad 1x_4$ ← all coefficients should add to 1 (account for all prob. of movements)

then matrix A:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

// left off here on Monday

system:



then $\vec{y} = A\vec{x} = T(\vec{x})$

↑ transformation matrix, A

let's look at the structure of A

let page i = resulting page

page j = starting page

let c_j = # of links from page j

$c_1 = 2$
$c_2 = 2$
$c_3 = 1$
$c_4 = 1$

proportion of users that start @ j, end @ i

$= \frac{x_j}{c_j}$ ← assuming initial population of x_j got distributed equally to c_j

equal probability

then ijth entry of matrix A is $\frac{1}{c_j}$ if $\exists$ link $j \to i$

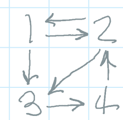
else if $\nexists$ link $\Rightarrow$ probability $= 0$

Equilibrium Distribution

construct the transformation s.t. $A\vec{x} = \vec{x}$

i.e., the distribution after movement is exactly same as before

system:



$y_1 = x_1$, etc.

$$\begin{array}{cccc|l} x_1 & x_2 & x_3 & x_4 & \\ & \frac{1}{2}x_2 & & & = x_1 \\ \frac{1}{2}x_1 & & & + x_4 & = x_2 \\ \frac{1}{2}x_1 & + \frac{1}{2}x_2 & & & = x_3 \\ & & x_3 & & = x_4 \end{array}$$

next, set all equations = 0

$$\left|\begin{array}{l} \frac{1}{2}x_1 + \frac{1}{2}x_2 = x_3 \\ x_3 = x_4 \end{array}\right|$$

$$\downarrow$$

$$\left|\begin{array}{rrrrl} -x_1 & +\frac{1}{2}x_2 & & & = 0 \\ \frac{1}{2}x_1 & -x_2 & & +x_4 & = 0 \\ \frac{1}{2}x_1 & +\frac{1}{2}x_2 & -x^3 & & = 0 \\ & & x^3 & -x^4 & = 0 \end{array}\right|$$

$\downarrow$ as an augmented matrix, M:

$$\left[\begin{array}{cccc|c} -1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \longrightarrow rref(M) = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\uparrow$ let $x_4 =$ arbitrary t

$x_1 - \frac{2}{3}x^4 = 0$

$\therefore x_1 = \frac{2}{3}x^4 \rightarrow x_1 = \frac{2}{3}t$

$x_2 = \frac{4}{3}t$

$x_3 = t$

$x_4 = t$

To find equilibrium vector, recall that sum of probabilities $= 1$

recall $\vec{x}$ was defined as the distribution vector.

Let $\vec{x}_{equ.}$ = equilibrium vector

$$\vec{x}_{equ} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$x_1 + x_2 + x_3 + x_4 = 1 \leftarrow$ substitute and solve for t

$$\underbrace{\frac{2}{3}t + \frac{4}{3}t}_{2t} + \underbrace{t + t}_{2t} = 1$$

$2t + 2t = 1$

$t = \frac{1}{4}$

$$= \begin{bmatrix} \frac{2}{3}\cdot\frac{1}{4} \\ \frac{4}{3}\cdot\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Check:

$$\frac{1}{6} + \frac{1}{3}\cdot\frac{2}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}}$$

$= \frac{3}{6} + \frac{1}{2}$

$= \frac{1}{2} + \frac{1}{2} = 1$ ✓

Definition:

- vector $\vec{x}$ in $\mathbb{R}^n$ is a distribution vector if its components add to 1 and all components are ≥ 0
- square matrix, $A :=$ transition matrix if $\forall$ components are distribution vectors, i.e., all entries are ≥ 0 and each set of column vectors add to 1
- if A is transition matrix and $\vec{x}$ is distribution vector then $A\vec{x}$ will be a distribution vector as well

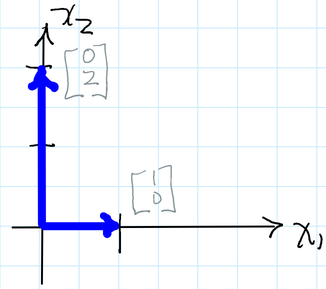
• if A is transition matrix and $\vec{x}$ is distribution vector then $A\vec{x}$ will be a distribution vector as well

see that we showed that A meets the conditions of a transition matrix
and $\vec{x}_{equ}$ " distribution vector

§2.2 Linear Transformations in Geometry

in previous section, saw that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ results in a 90° CCW rotation

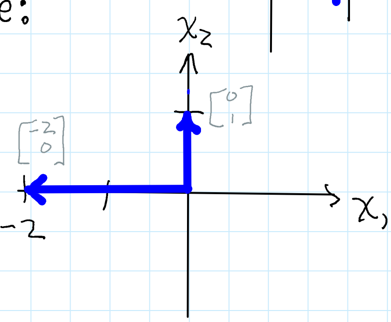
given vectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ graph as:



recall: apply transformation from above:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0(1)-1(0) \\ 1(1)+0(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0(0)-1(2) \\ 1(0)+0(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

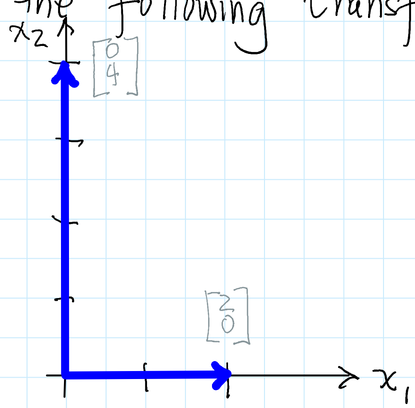


now, look at the results as the following transformations are applied:

$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

scalar

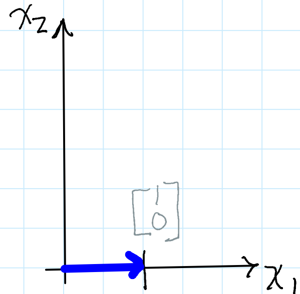
I_2



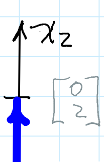
$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

eliminates vertical vector

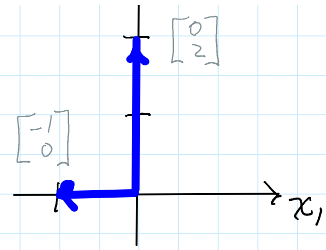
horizontal vector same



$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflection about vertical axis



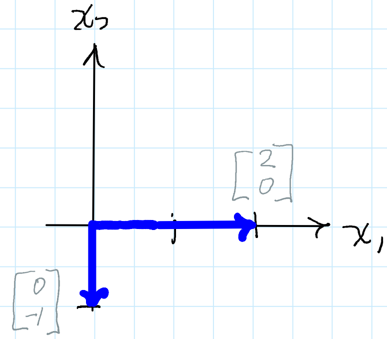
$$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$ reflection about vertical axis



$$D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$ CW rotation of 90°

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0(1) + 1(0) \\ -1(1) + 0(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0(0) + 1(2) \\ -1(0) + 0(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



more transformation exploration next time